Critical exponents of the four-state Potts model

1997 J. Phys. A: Math. Gen. 30 8785

(http://iopscience.iop.org/0305-4470/30/24/036)

J. Phys. A: Math. Gen. **30** (1997) 8785–8786. Printed in the UK PII: S0305-4470(97)86388-3

COMMENT

Critical exponents of the four-state Potts model

Richard J Creswick and Seung-Yeon Kim

Department of Physics and Astronomy, University of South Carolina, Columbia, SC 29208, USA

Received 29 July 1997

Abstract. The critical exponents of the four-state Potts model are derived directly from the exact expressions for the latent heat, the spontaneous magnetization and the correlation length at the transition temperature of the model in the limit $q \to 4^+$.

The q-state Potts model [1] in two dimensions exhibits a rich variety of critical behaviour. For $q = 2$ (Ising model), 3 and 4 there is a second-order phase transition, while for $q > 4$ the transition is first order [2]. With the exception of the Ising model, exact solutions for arbitrary q are not known. However, many exact results including the critical temperatures [1] and critical exponents [3–6, 8, 9], latent heat (for $q > 4$) [2] and spontaneous magnetization ($q > 4$) [7] at T_c are known. Most recently, Buffernoir and Wallon [10] have calculated the correlation length at T_c ($q > 4$), and it is on this paper that we wish to comment.

The latent heat at T_c is given by [2]

$$L = 2(1 + q^{-\frac{1}{2}}) \tanh \frac{\theta}{2} \prod_{n=1}^{\infty} (\tanh n\theta)^2 \qquad \text{for } q > 4 \tag{1}$$

where θ is defined by $2\cosh\theta = q^{\frac{1}{2}}$. The spontaneous magnetization at T_c is [7]

$$M_0 = \prod_{n=1}^{\infty} \frac{1 - x^{2n-1}}{1 + x^{2n}} \qquad \text{for } q > 4 \tag{2}$$

where $x = \mathrm{e}^{-2\theta}$. The correlation length, ξ, at T_c is given by [10]

$$\xi^{-1} = 2 \ln \frac{\cosh \frac{3}{4}\theta}{\cosh \frac{1}{4}\theta} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{e}^{-n\theta} \left(\sinh n \frac{\theta}{2} \right) (\tanh n\theta) \qquad \text{for } q > 4. \tag{3}$$

It is possible to view the four-state Potts model as the critical endpoint of a sequence of models with $q > 4$ which exhibit a first-order phase transition. At T_c and in the neighbourhood of $q \gtrsim 4$, the latent heat, spontaneous magnetization and correlation length are given by

$$L \sim 2\pi (1 + q^{-\frac{1}{2}}) \mathrm{e}^{-\frac{\pi^2}{2}(q-4)^{-\frac{1}{2}}} \tag{4}$$

$$M_0 \sim 2\mathrm{e}^{-\frac{\pi^2}{8}(q-4)^{-\frac{1}{2}}} \tag{5}$$

and

$$\xi^{-1} \sim \frac{8}{\sqrt{2}} e^{-\pi^2 (q-4)^{-\frac{1}{2}}}. \tag{6}$$

Each of these quantities exhibits the same essential singularity as $q \to 4^+$.

The central assumption underlying scaling theory is that the non-analytic behaviour of all thermodynamic functions such as the latent heat and spontaneous magnetization are tied to the non-analytic behaviour of the correlation length. Equations (4)–(6) are a clear example of this if we regard q as a continuous parameter. Further, if we define the critical exponents $y_t = 1/\nu$ and y_h in the usual way, then the scaling theory requires that

$$L \sim \xi^{-d+y_t} = \xi^{-\frac{1-\alpha}{\nu}} \tag{7}$$

and

$$M_0 \sim \xi^{-d+y_h} = \xi^{-\frac{\beta}{\nu}}. \tag{8}$$

If we compare equations (7) and (8) with equations (4) and (5), we see that $y_t = \frac{3}{2}$ and $y_h = \frac{15}{8}$, which are the known values for the critical exponents of the four-state Potts model defined in the usual way in terms of temperature.

Thus the Potts model offers a remarkable illustration of the scaling hypothesis even when there is an essential singularity in the correlation length rather than the usual power law. Finally, the singularities in $M_0(T, q)$ and $L(T, q)$ in the neighbourhood of the critical point $(T_c, 4)$ are determined by the same critical exponents (y_t, y_h) and the singularity in $\xi(T, q)$ whether one approaches the critical point for $q = 4$ and $T \to T_c$, or, $T = T_c$ and $q \to 4$.

The critical properties of the four-state Potts model have been studied extensively as the limiting case of a sequence ($q \leqslant 4$) of models with continuous phase transitions. As is often the case, the limit of such a sequence, $q = 4$, exhibits strong corrections to scaling. It is, therefore, of interest to approach this problem from the opposite side, and regard the four-state Potts model as the limit of a sequence of models ($q > 4$) with a discontinuous, or first-order, transition. As $q \to 4^+$, the latent heat and spontaneous magnetization at T_c vanish, and the correlation length diverges. We have shown that by applying simple scaling arguments to exact calculations of L, M_0, and ξ at T_c, one can derive the exact critical exponents and that they agree with those obtained for $q \leqslant 4$.

References

[1] Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
[2] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
[3] den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
[4] Nienhuis B, Riedel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L189
[5] Pearson R B 1980 *Phys. Rev.* B **22** 2579
[6] Black J and Emery V J 1981 *Phys. Rev.* B **23** 429
[7] Baxter R J 1982 *J. Phys. A: Math. Gen.* **15** 3329
[8] den Nijs M P M 1983 *Phys. Rev.* B **27** 1674
[9] Dotsenko Vl S 1984 *Nucl. Phys.* B **235** 54
Dotsenko Vl S and Fateev V A 1984 *Nucl. Phys.* B **240** 312
[10] Buffernoir E and Wallon S 1993 *J. Phys. A: Math. Gen.* **26** 3045